

CMC-1 TRINOIDS IN HYPERBOLIC 3-SPACE AND METRICS OF CONSTANT CURVATURE ONE WITH CONICAL SINGULARITIES ON THE 2-SPHERE

S. FUJIMORI, Y. KAWAKAMI, M. KOKUBU, W. ROSSMAN, M. UMEHARA,
AND K. YAMADA

ABSTRACT. CMC-1 trinoids (i.e. constant mean curvature one immersed surfaces of genus zero with three regular embedded ends) in hyperbolic 3-space H^3 are irreducible generically, and the irreducible ones have been classified. However, the reducible case has not yet been fully treated, so here we give an explicit description of CMC-1 trinoids in H^3 that includes the reducible case.

1. INTRODUCTION

Let H^3 denote the hyperbolic 3-space of constant sectional curvature -1 .

A *CMC-1 trinoid* in H^3 is a complete immersed constant mean curvature one surface of genus zero with three regular embedded ends. There are CMC-1 trinoids with horospherical ends (i.e. regular embedded ends which are asymptotic to a horosphere). However, an irreducible trinoid admits only catenoidal ends. The last two authors [9] gave a classification of those CMC-1 trinoids in H^3 that are irreducible. In particular, they showed that the moduli space of irreducible CMC-1 trinoids in H^3 (i.e. the quotient space of such immersions by the rigid motions of H^3) corresponds to a certain open dense subset of the set of irreducible spherical (i.e. constant curvature 1) metrics with three conical singularities (see Section 2). The paper [9] also investigated the reducible case, but had not obtained a complete classification there.

After that, Bobenko, Pavlyukevich, and Springborn [1] developed a representation formula for CMC-1 surfaces in H^3 in terms of holomorphic spinors and derived explicit parametrizations for irreducible CMC-1 trinoids in H^3 in terms of hypergeometric functions. The crucial step in [1] was a direct reduction of the ordinary differential equation that produces CMC-1 trinoids into a Fuchsian differential equation with three regular singularities, and we call this *BPS-reduction*. On the other hand, Daniel [2] gave an alternative proof of the classification theorem for irreducible CMC-1 trinoids, by applying Riemann's classical work on minimal surfaces in \mathbf{R}^3 bounded by three straight lines.

After the work [9] on the irreducible case, Furuta and Hattori [4] gave a full classification of spherical metrics with three conical singularities, using a purely geometric method. Later, Eremenko [3] proved it using hypergeometric equations.

Date: August 16, 2011.

2010 Mathematics Subject Classification. Primary 53A10, 53A35; Secondary 53C42, 33C05.

Key words and phrases. constant mean curvature, spherical metrics, conical singularities, trinoids.

In this paper, using the argument in [3] and the BPS-reduction, we describe a complete classification of reducible CMC-1 trinoids in H^3 .

2. PRELIMINARIES

Let M^2 be a 2-manifold, and consider a CMC-1 immersion $f : M^2 \rightarrow H^3$. The existence of such an immersion implies orientability of M^2 . By the existence of isothermal coordinates, there is a unique complex structure on M^2 such that the metric ds_f^2 induced by f is conformal (i.e. ds_f^2 is Hermitian). In this situation, there exists a holomorphic immersion (called a *null lift* of f)

$$F : \tilde{M}^2 \rightarrow \mathrm{SL}(2, \mathbf{C})$$

defined on the universal cover \tilde{M}^2 of M^2 so that:

- F is a *null* holomorphic map, namely, $F_z := dF/dz$ is of rank less than 2 on each local complex coordinate $(U; z)$ of M^2 .
- $f \circ \pi = \hat{\pi} \circ F$, where $\pi : \tilde{M}^2 \rightarrow M^2$ is the covering projection and

$$\hat{\pi} : \mathrm{SL}(2, \mathbf{C}) \rightarrow H^3 = \mathrm{SL}(2, \mathbf{C})/\mathrm{SU}(2)$$

is the canonical projection.

Then there exist a meromorphic function g and a holomorphic 1-form ω on \tilde{M}^2 such that

$$(1) \quad F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega,$$

and the first fundamental form ds_f^2 of f satisfies

$$ds_f^2 = (1 + |g|^2)^2 |\omega|^2.$$

The second fundamental form of f is given by

$$h := -Q - \bar{Q} + ds_f^2 \quad (Q := \omega dg),$$

where the holomorphic 2-differential Q on M^2 is called the *Hopf differential* of f . The set of zeros of Q corresponds to the set of umbilics of f . We set

$$(2) \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

Since $\det(dF) = 0$, one can easily show via (1) that

$$(3) \quad g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}.$$

With $\pi_1(M^2)$ denoting the covering transformation group on the universal cover \tilde{M}^2 , for each $\tau \in \pi_1(M^2)$, there exists $\rho(\tau) \in \mathrm{SU}(2)$ such that

$$(4) \quad F \circ \tau = F\rho(\tau),$$

which gives a representation (i.e. a group homomorphism) $\rho : \pi_1(M^2) \rightarrow \mathrm{SU}(2)$ satisfying

$$(5) \quad g \circ \tau^{-1} = \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}} =: \rho(\tau) \star g,$$

for each $\tau \in \pi_1(M^2)$, where $\rho(\tau) = (a_{ij})_{i,j=1,2}$.

Definition 1. A representation $\rho : \pi_1(M^2) \rightarrow \mathrm{SU}(2)$ is called *reducible* if $\rho(\pi_1(M^2))$ is abelian and otherwise is called *irreducible*. A CMC-1 immersion $f : M^2 \rightarrow H^3$ is called *irreducible* (*reducible*) if the induced representation ρ is irreducible (reducible).

The meromorphic function (cf. [8])

$$G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}$$

is well-defined on M^2 , and is called the *hyperbolic Gauss map* of f .

We now consider a CMC-1 immersion f satisfying the following properties:

- (a) The metric ds_f^2 induced by f is complete and of finite total curvature.

By (a), there exists a closed Riemann surface \bar{M}^2 such that M^2 is bi-holomorphic to $\bar{M}^2 \setminus \{p_1, \dots, p_n\}$, where p_1, \dots, p_n are distinct points of \bar{M}^2 called the *ends* of f . Then, the Hopf differential Q has at most a pole at each of p_1, \dots, p_n .

Now, we suppose the second condition:

- (b) All the ends p_1, \dots, p_n of f are properly embedded, namely, there is a neighborhood U_j of p_j in \bar{M}^2 such that the restriction $f|_{U_j \setminus \{p_j\}}$ is a proper embedding, for each $j = 1, \dots, n$.

Then, the condition (b) implies that G has at most a pole at each end p_j ($j = 1, \dots, n$), namely, the ends p_1, \dots, p_n are all regular ends.

Definition 2 ([7]). Let \bar{M}^2 be a closed Riemann surface. Let $d\sigma^2$ be a conformal metric on $\bar{M}^2 \setminus \{p_1, \dots, p_n\}$, where p_1, \dots, p_n are distinct points. Then $d\sigma^2$ has a *conical singularity* of order μ_j at p_j if $\mu_j > -1$ and $d\sigma^2/|z|^{2\mu_j}$ is positive definite at p_j , where z is a local coordinate so that $z = 0$ at p_j . $2\pi(1 + \mu_j)$ is called the *conical angle* of $d\sigma^2$ at p_j .

We set $\bar{M}^2 = S^2$ and consider conformal metrics that have exactly three conical singularities at $0, 1, \infty$ on $S^2 = \mathbf{C} \cup \{\infty\}$. We denote by $\mathcal{M}_3(S^2)$ the set of such metrics having constant curvature 1 on $M^2 := \mathbf{C} \setminus \{0, 1\}$, namely, $\mathcal{M}_3(S^2)$ can be identified with the moduli space of conformal metrics of constant curvature 1 with three conical singularities. We fix a metric $d\sigma^2 \in \mathcal{M}_3(S^2)$, and then there exists a developing map

$$g : \tilde{M}^2 \rightarrow S^2 = \mathbf{C} \cup \{\infty\}$$

so that $d\sigma^2 = 4dg d\bar{g}/(1+|g|^2)^2$, where \tilde{M}^2 is the universal cover of $M^2 (= \mathbf{C} \setminus \{0, 1\})$. Then there is a representation ([9, (2.15) and Lemma 2.2])

$$(6) \quad \rho : \pi_1(M^2) \rightarrow \mathrm{SU}(2)$$

satisfying (5). The metric $d\sigma^2$ is called *irreducible* if ρ is irreducible.

We return to the previous situation of CMC-1 surfaces. Let K be the Gaussian curvature of the CMC-1 immersion f . Then

$$(7) \quad d\sigma_f^2 := (-K)ds_f^2 = \frac{4dg d\bar{g}}{(1+|g|^2)^2}.$$

This relation implies that $d\sigma_f^2$ has constant curvature 1 wherever $d\sigma_f^2$ is positive definite. Moreover ([8]),

$$ds_f^2 d\sigma_f^2 = 4Q\bar{Q}$$

implies that $d\sigma_f^2$ has a conical singularity at a zero q of Q , and the conical order of $d\sigma_f^2$ at q equals Q 's order there. The condition (a) implies that $d\sigma_f^2$ has also a conical singularity at each end p_j .

Definition 3. Let $f : M^2 \rightarrow H^3$ be a CMC-1 immersion satisfying conditions (a) and (b). Then f is called a *CMC-1 n -noid* if \bar{M}^2 is conformally equivalent to the 2-sphere S^2 . An end p of a CMC-1 n -noid is called *catenoidal* if Q has a pole of order 2 at p . A CMC-1 n -noid is called *catenoidal* if all ends are catenoidal.

Let f be a CMC-1 n -noid. When $n = 1$, f is congruent to the horosphere. When $n = 2$, f is congruent to a catenoid cousin or a warped catenoid cousin (cf. [6]).

So it is natural to consider the case $n = 3$. Since the three ends are embedded, the Osserman-type inequality ([8]) implies $\deg(G) = 2$. We call a CMC-1 3-noid a *trinoid* (or a CMC-1 trinoid). We denote by $\mathcal{M}_3(H^3)$ the set of congruence classes of trinoids. We now fix a trinoid f . As shown in [5], there are only two possibilities:

- (i) Q has poles of order 2 at p_1, p_2, p_3 .
- (ii) Q has at most poles of order 2 at p_1, p_2, p_3 , but at least one of the p_j has a pole of order 1.

As CMC-1 trinoids satisfying (i) are catenoidal, irreducible trinoids are catenoidal (see [9]). CMC-1 immersions satisfying (ii) have been classified in [5, Theorems 4.5–4.7]. So from now on we consider just the case (i). Without loss of generality we may assume $p_1 = 0, p_2 = 1, p_3 = \infty$. As mentioned above, the metric $d\sigma_f^2$ given by (7) has conical singularities at the zeros of Q and the three ends p_1, p_2, p_3 . We denote by $\beta_j (> -1)$ the order of $d\sigma_f^2$ at p_j , and by

$$B_j := \pi(1 + \beta_j) (> 0) \quad (j = 1, 2, 3)$$

the half of the conical angle of $d\sigma_f^2$ at p_j ($j = 1, 2, 3$). The group $\rho(\pi_1(M^2))$ is generated by three monodromy matrices ρ_1, ρ_2, ρ_3 which represent loops surrounding $z = 0, 1, \infty$. Each ρ_j ($j = 1, 2, 3$) has eigenvalues $-\exp(\pm iB_j)$. Then we have (cf. [9])

$$(8) \quad 2Q = \frac{c_3 z^2 + (c_2 - c_3 - c_1)z + c_1}{z^2(z-1)^2} dz^2,$$

where $c_j := -\beta_j(\beta_j + 2)/2$ does not vanish by (i) (i.e. $B_j \neq \pi$) for $j = 1, 2, 3$, and

$$(9) \quad \frac{(c_1)^2 + (c_2)^2 + (c_3)^2}{2} \neq c_1 c_2 + c_2 c_3 + c_3 c_1.$$

We denote by q_1, q_2 the two roots of the equation

$$(10) \quad c_3 z^2 + (c_2 - c_3 - c_1)z + c_1 = 0.$$

Since $c_3 \neq 0$, the Hopf differential Q has exactly two zeros at q_1 and q_2 . In fact, (9) is equivalent to the condition $q_1 \neq q_2$ (i.e. the discriminant of (10) does not vanish). As shown in [9], the condition (b) implies that G does not branch at the three ends $0, 1, \infty$, but has exactly two branch points at q_1, q_2 . Since G is of degree 2 and has the ambiguity of Möbius transformations, we may set (cf. [9])

$$(11) \quad G := z + \frac{(q_1 - q_2)^2}{2(2z - q_1 - q_2)}.$$

Take a solution $F : \tilde{M}^2 \rightarrow \mathrm{SL}(2, \mathcal{C})$ of the ordinary differential equation

$$(12) \quad dFF^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}.$$

If the image $\rho(\pi_1(M^2))$ of the representation ρ of F is conjugate to a subgroup of $\mathrm{SU}(2)$, then $f = \hat{\pi}(Fa)$ gives a CMC-1 trinoid for a suitable choice of $a \in \mathrm{SL}(2, \mathcal{C})$ (cf. (4)). We denote by $\mathcal{M}_{B_1, B_2, B_3}(H^3)$ (resp. $\mathcal{M}_{B_1, B_2, B_3}(S^2)$) the congruence classes of trinoids f satisfying (i) (resp. of the metrics $d\sigma^2$ of constant curvature 1) such that $d\sigma_f^2$ (resp. $d\sigma^2$) has conical angle $2B_j (\neq 2\pi)$ at each p_j .

Fact 1 ([9]). *For each $B_1, B_2, B_3 \in (0, \infty)$, $\mathcal{M}_{B_1, B_2, B_3}(H^3)$ (resp. $\mathcal{M}_{B_1, B_2, B_3}(S^2)$) consists of a unique irreducible element if it satisfies (9) (resp. no condition) and*

$$(13) \quad \cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2 \cos B_1 \cos B_2 \cos B_3 < 1.$$

Conversely, any irreducible trinoids (resp. any irreducible metrics in $\mathcal{M}_3(S^2)$) are so obtained.

In particular, there is a unique catenoidal trinoid f such that

- *the hyperbolic Gauss map G is given by (11),*
- *the Hopf differential Q is given by (8),*
- *$d\sigma_f^2$ has conical angle $2B_j$ at each end p_j .*

Figure 1, left (resp. right) is an irreducible trinoid (resp. a cutaway view of an irreducible trinoid) for $B_1 = B_2 = B_3 (= B)$ with $B < \pi$ (resp. $B > \pi$).

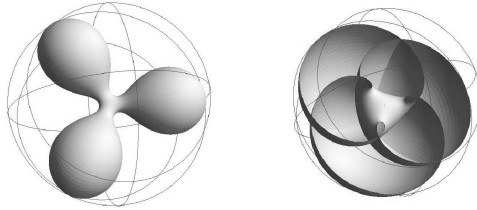


FIGURE 1. Trinoids with $B_1 = B_2 = B_3$.

Remark 1. Since the hyperbolic Gauss map G changes under a rigid motion of H^3 , the above trinoid f is uniquely determined without the ambiguity of isometries of H^3 (cf. [9, Appendix B]). After [9], Bobenko, Pavlyukevich and Springborn [1] gave a different proof, whose underlying idea also appears in the next section. Also, Daniel [2] gave an alternative proof of this fact (see the introduction).

For each B_j ($j = 1, 2, 3$) there exists a unique real number $\hat{B}_j \in [0, \pi]$ such that $\cos B_j = \cos \hat{B}_j$, since $\cos t = \cos(2\pi - t)$ for $t \in [0, 2\pi)$. By definition, it holds that $B_j \geq \hat{B}_j$. Without loss of generality, we may assume that $\hat{B}_1 \leq \hat{B}_2 \leq \hat{B}_3$. We now set $B'_1 := \hat{B}_1$, and for $j = 2, 3$,

$$B'_j := \begin{cases} \hat{B}_j & \text{if } \hat{B}_2 + \hat{B}_3 \leq \pi, \\ \pi - \hat{B}_j & \text{if } \hat{B}_2 + \hat{B}_3 > \pi. \end{cases}$$

Then we have that

$$(14) \quad 0 \leq B'_1 + B'_2, B'_1 + B'_3, B'_2 + B'_3 \leq \pi,$$

and the condition (13) is equivalent to

$$\cos^2 B'_1 + \cos^2 B'_2 + \cos^2 B'_3 + 2 \cos B'_1 \cos B'_2 \cos B'_3 < 1,$$

which is equivalent to the condition

$$\begin{aligned} & \cos \frac{B'_1 + B'_2 + B'_3}{2} \cos \frac{-B'_1 + B'_2 + B'_3}{2} \\ & \quad \times \cos \frac{B'_1 - B'_2 + B'_3}{2} \cos \frac{B'_1 + B'_2 - B'_3}{2} < 0. \end{aligned}$$

By (14), this then reduces to the condition

$$(15) \quad B'_1 + B'_2 + B'_3 > \pi.$$

The condition (13) (or equivalently (15)) implies $B_j \notin \pi\mathbf{Z}$ ($j = 1, 2, 3$), and is the same condition as in [9], [4] or [3] that there exists an irreducible metric in $\mathcal{M}_3(S^2)$ with three conical angles $2B_1, 2B_2, 2B_3$.

3. REDUCIBLE TRINOIDS

Let α be a 2×2 -matrix valued meromorphic 1-form on $\mathbf{C} \cup \{\infty\}$. Consider an ordinary differential equation

$$(16) \quad dEE^{-1} = \alpha,$$

which is called a *Fuchsian differential equation* if it admits only regular singularities. For example, the equation (12) with G, Q satisfying (11) and (8) is a Fuchsian differential equation with regular singularities at $z = 0, 1, \infty, q_1, q_2$. Let $p_1, \dots, p_n \in \mathbf{C} \cup \{\infty\}$ be the regular singularities of the equation (16). We denote by \tilde{M}^2 the universal cover of

$$M^2 := \mathbf{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}.$$

Then there exists a solution $E : \tilde{M}^2 \rightarrow \mathrm{GL}(2, \mathbf{C})$ of (16). Since α is defined on M^2 , there exists a representation $\gamma : \pi_1(M^2) \rightarrow \mathrm{GL}(2, \mathbf{C})$ such that $E \circ \tau = E\gamma(\tau)$. Let

$$\mathrm{GL}(2, \mathbf{C}) \ni a \mapsto [a] \in \mathrm{PGL}(2, \mathbf{C}) = \mathrm{PSL}(2, \mathbf{C})$$

be the canonical projection. Then

$$h_1 := -E_{12}/E_{11}, \quad h_2 := -E_{22}/E_{21}$$

satisfy (see (5) for the definition of \star)

$$h_i \circ \tau^{-1} = \gamma(\tau) \star h_i \quad (\tau \in \pi_1(M^2), i = 1, 2),$$

where $E = (E_{jk})_{j,k=1,2}$. Thus the functions h_i ($i = 1, 2$) induce a common group homomorphism $[\gamma] : \pi_1(M^2) \rightarrow \mathrm{PGL}(2, \mathbf{C})$ which is called the *monodromy representation* of the equation (16). In particular, the representation $[\rho]$ for F as in (12) is just the monodromy representation.

Definition 4. Let $r(z), s(z)$ be meromorphic functions on $\mathbf{C} \cup \{\infty\}$ and

$$(17) \quad X'' + rX' + sX = 0$$

be an ordinary differential equation with regular singularities at $z = p_1, \dots, p_n$, where $X' = dX/dz$. Then there exists a pair of solutions $w_1, w_2 : \tilde{M}^2 \rightarrow \mathbf{C}$ which are linearly independent, and $\{w_1, w_2\}$ is called a fundamental system of solutions.

There exists a representation $\gamma : \pi_1(M^2) \rightarrow \mathrm{GL}(2, \mathbf{C})$ for each fundamental system $\{w_1, w_2\}$, such that

$$(w_1 \circ \tau, w_2 \circ \tau) = (w_1, w_2)\gamma(\tau),$$

where (w_1, w_2) is a row vector. As a monodromy of the function $-w_2/w_1$, the induced homomorphism $[\gamma] : \pi_1(M^2) \rightarrow \mathrm{PGL}(2, \mathbf{C})$ is called the *monodromy representation* of the equation (17).

To give a complete classification of trinoids, the following reduction given in [1] is crucial: Let F be a null lift of the catenoidal trinoid f whose hyperbolic Gauss map G and Hopf differential Q are given by (11) and (8), respectively. In the expression (12), we can write

$$\begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} = \begin{pmatrix} \mathbb{P}_1 \mathbb{P}_2 & (\mathbb{P}_1)^2 \\ -(\mathbb{P}_2)^2 & -\mathbb{P}_1 \mathbb{P}_2 \end{pmatrix} dz,$$

where $\mathbb{P}_i := \frac{p_i^0}{z} + \frac{p_i^1}{z-1} + p_i^\infty$ and p_i^0, p_i^1, p_i^∞ ($i = 1, 2$) are constants depending only on B_1, B_2, B_3 . In [1], the matrix $\Phi := D^{-1}F$ is defined by

$$D := \sqrt{z-1} \begin{pmatrix} \mathbb{P}_1 & \alpha_1 z + \beta_1 \\ -\mathbb{P}_2 & \alpha_2 z + \beta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{k}{z(z-1)} & \frac{1}{z-1} \end{pmatrix} \begin{pmatrix} \vartheta & 0 \\ 1 & 1 \end{pmatrix},$$

where α_j, β_j ($j = 1, 2$), k and ϑ are all real constants depending only on B_1, B_2, B_3 . Then there exist 2×2 matrices A_0, A_1 with real coefficients such that

$$(18) \quad d\Phi\Phi^{-1} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} \right) dz.$$

We call (18) the *BPS-reduction* of (12). (This reduction does not work if f has a horospherical end, but such trinoids would be in the case (ii) mentioned before.) By (4), it holds that $\Phi \circ \tau = \Phi\rho(\tau)$ for each $\tau \in \pi_1(M^2)$. Obviously (18) has three regular singularities at $z = 0, 1, \infty$. Since A_0 and A_1 are both constant real matrices, it is well-known that there exist real numbers a, b, c such that the monodromy representation of the ordinary differential equation (called the *hypergeometric equation*)

$$(19) \quad z(1-z)X'' + (c - (a+b+1)z)X' - abX = 0$$

is conjugate to that of (18) (i.e. $[\rho]$). On the other hand, if we express F as in (2), then $X = F_{11}, F_{12}$ satisfy the ordinary differential equation (cf. [5, p. 32])

$$(20) \quad X'' - (\log(\hat{Q}/G'))'X' + \hat{Q}X = 0,$$

where $Q = \hat{Q}(z)dz^2$ and $G' = dG/dz$. Thus the monodromy representation of (20) with respect to (F_{11}, F_{12}) is equal to that of F . In particular, the monodromy representation of (20) is conjugate to that of (19). Hence, these two ordinary differential equations have the same *exponent* (i.e. the difference of the two solutions of the indicial equation) at each regular singularity. Since (20) has the exponent $B_1/\pi, B_2/\pi, B_3/\pi$ at $z = 0, 1, \infty$, respectively, we have

$$\begin{aligned} \pm B_1 &= \pi(1-c), & \pm B_2 &= \pi(a-b), \\ \pm B_3 &= \pi(c-a-b), \end{aligned}$$

which is the same set of relations as in [3, (4)]. This implies the classification of catenoidal trinoids reduces to that of metrics in $\mathcal{M}_3(S^2)$. In particular, the

classification results for reducible metrics in $\mathcal{M}_3(S^2)$ given in Furuta-Hattori [4] and Eremenko [3, Theorem 2] yield the following assertion.

Theorem. *Suppose B_1/π is an integer, and $B_j \neq \pi$ ($j = 1, 2, 3$). Then $\mathcal{M}_{B_1, B_2, B_3}(H^3)$ (resp. $\mathcal{M}_{B_1, B_2, B_3}(S^2)$) is non-empty if and only if B_1, B_2, B_3 satisfy (9) (resp. no condition) and one of the following two conditions:*

- (C₁) $B_2, B_3 \notin \pi\mathbf{Z}$, but either $|B_2 - B_3|/\pi$ or $(B_2 + B_3)/\pi$ is an integer m of opposite parity from B_1/π , and $\pi m \leq B_1 - \pi$. In this case, $\mathcal{M}_{B_1, B_2, B_3}(H^3)$ (resp. $\mathcal{M}_{B_1, B_2, B_3}(S^2)$) is 1-dimensional.
- (C₂) $B_2, B_3 \in \pi\mathbf{Z}$, and $(B_1 + B_2 + B_3)/\pi$ is odd, and each of B_1, B_2, B_3 is less than the sum of the others. In this case, $\mathcal{M}_{B_1, B_2, B_3}(H^3)$ (resp. $\mathcal{M}_{B_1, B_2, B_3}(S^2)$) is 3-dimensional.

Corollary 1. *A catenoidal trinoid f is irreducible if and only if $B_1/\pi, B_2/\pi, B_3/\pi$ are all non-integers, and f is reducible if and only if at least one of $B_1/\pi, B_2/\pi, B_3/\pi$ is an integer.*

Proof. A trinoid f is irreducible if the representation ρ as in (4) is irreducible. The representation ρ coincides with that of the corresponding metric in $\mathcal{M}_3(S^2)$. The corresponding assertion for metrics in $\mathcal{M}_3(S^2)$ is proved in [9, Lemma 3.1]. \square

Remark 2. Reducibility is equivalent to at least one of $B_1/\pi, B_2/\pi, B_3/\pi$ being an integer. This cannot be proved purely algebraically, as there are diagonal matrices ρ_1, ρ_2, ρ_3 in $\mathrm{SU}(2)$ with $\rho_1\rho_2\rho_3 = \mathrm{id}$ so that no eigenvalues of ρ_1, ρ_2, ρ_3 are ± 1 .

Remark 3. Eremenko [3, Theorem 2] asserts the uniqueness of $d\sigma^2 \in \mathcal{M}_{B_1, B_2, B_3}(S^2)$ with prescribed conical angles. This is correct in the irreducible case, but if $B_1 \in \pi\mathbf{Z}$, then the metric has a nontrivial deformation preserving its conical angles: A metric $d\tau^2 \in \mathcal{M}_3(S^2)$ has the same conical angles as those of $d\sigma^2$ if and only if each developing map of $d\tau^2$ is given by $k = a \star h$ for suitable $a \in \mathrm{SL}(2, \mathbf{C})$, where h is a developing map of $d\sigma^2$. So $\mathcal{M}_{B_1, B_2, B_3}(S^2)$ can be identified with the set ([9])

$$\{\hat{\pi}(a); a(\mathrm{Im}\rho)a^{-1} \subset \mathrm{SU}(2), a \in \mathrm{SL}(2, \mathbf{C})\} \subset H^3,$$

where $\hat{\pi} : \mathrm{SL}(2, \mathbf{C}) \rightarrow H^3$ is the canonical projection and $\mathrm{Im}\rho$ is the image of ρ as in (6). Then $\mathcal{M}_{B_1, B_2, B_3}(S^2) = H^3$ if B_j/π ($j = 1, 2, 3$) are all integers, and $\mathcal{M}_{B_1, B_2, B_3}(S^2)$ is a geodesic line in H^3 if one of B_j/π ($j = 1, 2, 3$) is not an integer (cf. [9]). A metric $d\sigma^2$ in $\mathcal{M}_{B_1, B_2, B_3}(S^2)$ is called *symmetric* if the metric is invariant under an anti-holomorphic involution. We denote by $\hat{\mathcal{M}}_{B_1, B_2, B_3}(S^2)$ the subset consisting of symmetric metrics in $\mathcal{M}_{B_1, B_2, B_3}(S^2)$. If B_j/π ($j = 1, 2, 3$) are all integers, $\hat{\mathcal{M}}_{B_1, B_2, B_3}(S^2)$ consists of a hyperbolic plane in H^3 ([9]). If one of B_j/π ($j = 1, 2, 3$) is not an integer, $\hat{\mathcal{M}}_{B_1, B_2, B_3}(S^2)$ coincides with $\mathcal{M}_{B_1, B_2, B_3}(S^2)$. A metric $d\sigma^2$ in $\hat{\mathcal{M}}_{B_1, B_2, B_3}(S^2)$ with conical angles $2B_1, 2B_2$ and $2B_3$ can be regarded as a doubling of the generalized spherical triangle with angles B_1, B_2 and B_3 . Using this, Furuta-Hattori [4] gave two operations in $\hat{\mathcal{M}}_3(S^2)$ for distinct $\{i, j, k\} = \{1, 2, 3\}$:

$$\begin{aligned} (B_i, B_j, B_k) &\mapsto (B_i + \pi, B_j + \pi, B_k), \\ (B_i, B_j, B_k) &\mapsto (\pi - B_i, B_j + \pi, B_k), \end{aligned}$$

with the second operation allowed only when $B_i < \pi$. The first operation is attaching a closed hemisphere in S^2 to the edge $B_i B_j$ of the spherical triangle $\triangle B_i B_j B_k$. The second operation is attaching a geodesic 2-gon of equi-angles $\pi - B_i$ to the

edge $B_i B_j$ so that the initial vertex B_i becomes an interior point of an edge of the new triangle. Conditions (C_1) and (C_2) are invariant under these two operations. Moreover, the three angles (B_1, B_2, B_3) satisfying conditions (C_1) and (C_2) are obtained from a given initial data (B'_1, B'_2, B'_3) by these two operations. Furuta-Hattori proved this using a geometric argument. On the other hand, Eremenko found (C_1) and (C_2) from the viewpoint of hypergeometric equations. We remark that spherical triangles of arbitrary angles $B_1, B_2, B_3 \in (0, \infty)$ were investigated by Felix Klein in 1933 (see the end of [9]). The trinoid shown in Figure 2 is not symmetric, although there does exist a symmetric trinoid with the same conical angles and dihedral symmetry.

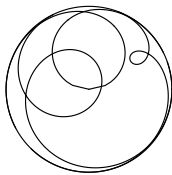


FIGURE 2. A profile curve of a non-symmetric trinoid with $B_1 = B_2 = B_3 = 3\pi$. (The outer circle represents the ideal boundary of H^3 .)

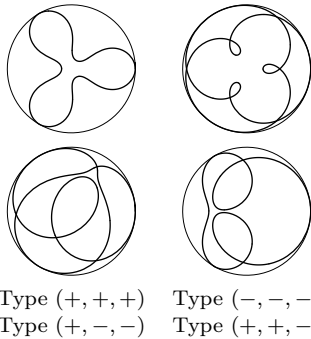


FIGURE 3. Profile curves of trinoids of different types.

Finally, we group the surfaces by the signatures of c_1, c_2, c_3 . For example, a trinoid f is said to be of type $(+, +, +)$ if c_1, c_2, c_3 are all positive, and of type $(-, +, +)$ if one of c_1, c_2, c_3 is negative and the other two are positive, etc. As remarked in [6], by numerical experiment, it seems that the four types $(+, +, +)$, $(-, +, +)$, $(-, -, +)$ and $(-, -, -)$ have distinct regular homotopy types (see Figure 3). Surfaces of type $(+, +, +)$ have absolute total curvature less than 8π , and it seems that only surfaces in this class can be embedded.

Acknowledgements. The authors thank Masaaki Yoshida and Yoshishige Haraoka for valuable comments, and for imparting to us the charm of the hypergeometric equation. They also thank Alexander Eremenko for valuable comments.

REFERENCES

- [1] A.I. Bobenko, T.V. Pavlyukevich and B.A. Springborn, *Hyperbolic constant mean curvature one surfaces: Spinor representation and trinoids in hypergeometric functions*, Math. Z. **245** (2003), 63–91.
- [2] B. Daniel, *Minimal disks bounded by three straight lines in Euclidean space and trinoids in hyperbolic space*, J. Differential Geom. **72** (2006), 467–508.
- [3] A. Eremenko, *Metrics of positive curvature with conic singularities on the sphere*, Proc. of Amer. Math. Soc. **132** (2004), 3349–3355.
- [4] M. Furuta and Y. Hattori, *2-dimensional singular spherical space forms*, manuscript, 1998.
- [5] W. Rossman, M. Umehara and K. Yamada, *Mean curvature 1 surfaces in hyperbolic 3-space with low total curvature I*, Hiroshima Math. J. **34** (2004), 21–56.
- [6] ———, *Period problems for mean curvature one surfaces in H^3* , Surveys on Geometry and Integrable systems, Advanced studies in Pure Mathematics **51** (2008), 347–399.
- [7] M. Troyanov, *Preserving curvature on compact surfaces with conical singularities*, Trans. of Amer. Math. Soc. **324** (1991), 793–820.
- [8] M. Umehara and K. Yamada, *A duality on CMC-1 surfaces in hyperbolic space, and a hyperbolic analogue of the Osserman inequality*, Tsukuba J. Math. **21** (1997), 229–237.
- [9] ———, *Metrics of constant curvature one with three conical singularities on the 2-sphere*, Illinois J. Math. **44** (2000), 72–94.

(Shoichi Fujimori) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, OKAYAMA UNIVERSITY, OKAYAMA 700-8530

E-mail address: fujimori@math.okayama-u.ac.jp

(Yu Kawakami) GRADUATE SCHOOL OF SCIENCE AND ENGINEERING, YAMAGUCHI UNIVERSITY, YAMAGUCHI, 753-8512

E-mail address: y-kwakami@yamaguchi-u.ac.jp

(Masatoshi Kokubu) DEPARTMENT OF MATHEMATICS, SCHOOL OF ENGINEERING, TOKYO DENKI UNIVERSITY, TOKYO 101-8457

E-mail address: kokubu@cck.dendai.ac.jp

(Wayne Rossman) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOBE UNIVERSITY, ROKKO, KOBE 657-8501

E-mail address: wayne@math.kobe-u.ac.jp

(Masaaki Umehara) DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO 152-8552

E-mail address: umehara@is.titech.ac.jp

(Kotaro Yamada) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO 152-8551

E-mail address: kotaro@math.titech.ac.jp